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THE PRESSURIZED HOLLOW SPHERE PROBLEM IN FINITE ELASTOSTATICS FOR A CLASS OF COMPRESSIBLE MATERIALS†

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Abstract—This investigation is concerned with the problem of a hollow sphere subjected to uniform internal and external pressure within the equilibrium theory of finite elasticity. The sphere is composed of homogeneous, isotropic, *compressible* materials of special type, namely harmonic materials. Explicit closed-form solutions for the deformation and stress fields are obtained. The true stress distribution, expressed as a function of the undeformed coordinates, is shown to be essentially independent of material properties. The two cases of internal pressure only, and external pressure only, are examined in detail. In the former case, there is a critical value of the applied pressure at which the maximum hoop stress in the sphere, occurring at the inner surface, becomes unbounded. Results appropriate for thin shells are also obtained. For the case of external pressure only, a critical value of the applied pressure exists for which the cavity closes. The maximum hoop stress does *not* always occur at the cavity wall. For nearly solid spheres, or equivalently, for a cavity in an unbounded medium, explicit results are provided for the corresponding *stress concentration factor*. For sufficiently small values of applied pressures, all the foregoing results coincide with those of classical linear isotropic elastostatics.

1. INTRODUCTION

In this paper we examine the finite elastostatic deformation of a hollow sphere subjected to uniform internal and external pressure. The resulting deformation and stress fields are found to bear some similarities, as well as certain striking differences, to the corresponding quantities in the infinitesimal theory of elasticity.

We consider a sphere composed of homogeneous, isotropic, *compressible* materials of special type, namely the harmonic materials introduced by John[1]. Harmonic materials are known to simplify the nonlinear partial differential equations governing finite elasticity (see, e.g. [1–5]) and are thus an attractive choice for constitutive models when exact analytical solutions are desired.

The corresponding problem for an *incompressible* material has been previously considered by Green and Shield[6] (see also[7], Section 3.10). In this case, the incompressibility constraint immediately yields an explicit expression for the (spherically symmetric) deformation field; the hydrostatic pressure field occurring in the constitutive law is then obtained from equilibrium by integration[6, 7]. Such simplification does not occur for compressible materials.

In the next section, we recall briefly some preliminary results from the nonlinear equilibrium theory for a special class of compressible materials, the *harmonic* materials of John[1]. In Section 3, the problem of a hollow sphere, composed of a harmonic material, and subjected to uniform internal and external pressure, is analyzed. Explicit closed form solutions for the radial and hoop stresses are obtained (see eqns (3.13) below) which are similar in structure to the classical results of linear elasticity theory[8]. Upon linearization, our results become identical to those of the linear theory.

The main features of the results are described in Section 4. Firstly, we observe that the Cauchy stress (true stress) distribution in the sphere, expressed as a function of the undeformed coordinates, is (essentially) independent of the constitutive properties of harmonic materials. Recall that the stresses given by linear elasticity theory[8] are similar in this respect. It is convenient to divide the subsequent discussion into the two cases of

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internal pressure only ($p_0 = 0, p_i \neq 0$) and external pressure only ($p_i = 0, p_0 \neq 0$). In the former case, we find that there is a critical value p_∞ of applied pressure (see eqn 4.2) for which the hoop stress at the inner surface becomes *unbounded*. This is a surprising result, since there are no geometric or load discontinuities in the problem. For values of $p_i < p_\infty$ we demonstrate the existence of a (bounded) solution of the problem; furthermore, the maximum hoop stress is shown to always occur at the inner surface. Finally, we derive an expression for the hoop stress for the case of a thin shell (see eqn 4.6).

The situation in the case of external loading only is quite different, in that unbounded stresses do not arise. We show that there is a critical value of applied external pressure $p_0 (p_0 = 2\mu)$ at which the spherical cavity closes. Moreover, the maximum hoop stress does *not* always occur at the inner surface.† For a sufficiently large value of the pressure, the location of the maximum hoop stress departs from the inner surface. As the pressure is further increased, this point continues to move towards the outer boundary. Results for the corresponding *stress concentration factor* are provided. Section 4 concludes with a brief discussion of the case of combined loading.

It should be noted that stability considerations, such as buckling, are not addressed in this investigation.‡

2. PRELIMINARIES

Consider a body occupying a region $\mathscr{R}_0$ in its unstressed state. A deformation of the body is described by a sufficiently smooth and invertible transformation $\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x})$ which maps $\mathscr{R}_0$ onto the region $\mathscr{R}$ occupied by the deformed body. The deformation gradient tensor $\mathbf{F}$ and the Jacobian determinant J are given by

$$\mathbf{F} = \nabla \hat{\mathbf{y}}(\mathbf{x}), J = \det \mathbf{F} > 0 \text{ on } \mathscr{R}_0, \tag{2.1}$$

and according to the polar decomposition theorem $\mathbf{F}$ admits the unique representation

$$\mathbf{F} = \mathbf{R}\mathbf{U}. \tag{2.2}$$

Here the stretch tensor $\mathbf{U}$ is symmetric and positive definite while the rotation tensor $\mathbf{R}$ is proper orthogonal. Moreover the principal stretches λ_1, λ_2, $\lambda_3 (> 0)$ of the deformation are the eigenvalues of $\mathbf{U}$, while the principal invariants may be taken to be

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, i_3 = \lambda_1\lambda_2\lambda_3. \tag{2.3}$$

Next, let $\boldsymbol{\sigma}(\mathbf{x})$ and $\boldsymbol{\tau}(\mathbf{y})$ be the first Piola–Kirchhoff and Cauchy stress tensor fields, respectively,

$$\boldsymbol{\sigma} = J\boldsymbol{\tau}(\mathbf{F}^{-1})^T. \tag{2.4}$$

The conditions for equilibrium in the absence of body force are

$$\operatorname{div} \boldsymbol{\tau}(\mathbf{y}) = \mathbf{0}, \boldsymbol{\tau} = \boldsymbol{\tau}^T \text{ on } \mathscr{R}, \tag{2.5}$$

while the traction $\mathbf{t}$ on a surface in the deformed body with unit outward normal $\mathbf{n}$ is given by

$$\mathbf{t} = \boldsymbol{\tau}\mathbf{n}. \tag{2.6}$$

Finally let W be the elastic potential of the compressible, homogeneous, isotropic, hyperelastic solid at hand. Then W depends on position in $\mathscr{R}_0$ exclusively through the invariants i_1, i_2, i_3 so that the Piola–Kirchhoff stress tensor accompanying the deformation

†This feature also occurs in the problem of a pressurized *incompressible* sphere composed of a Mooney–Rivlin material (see[7], p. 108).

‡See, however, Sensenig[10] for initial steps towards such an analysis.

is given by

$$\sigma = \left(\frac{\partial W}{\partial i_1} + i_1\frac{\partial W}{\partial i_2}\right)\mathbf{R} - \frac{\partial W}{\partial i_2}\mathbf{F} + i_3\frac{\partial W}{\partial i_3}(\mathbf{F}^T)^{-1}. \tag{2.7}$$

We turn now to the particular class of *harmonic materials* introduced by John[1] for which the elastic-potential has the form

$$W = 2\mu\{\mathscr{H}(i_1) - i_3\}, \mathscr{H}(3) = 1, \mathscr{H}'(3) = 1, \mathscr{H}''(3) > 0. \tag{2.8}\dagger$$

The constitutive relation (2.7) specializes in this case to

$$\sigma = 2\mu\{\mathscr{H}'(i_1)\mathbf{R} - i_3(\mathbf{F}^T)^{-1}\}. \tag{2.9}$$

In general, certain additional restrictions on the constitutive function $\mathscr{H}$ must be imposed in order to ensure a physically reasonable response of the material to pure homogeneous deformations.

For our purposes here it is essential that the material behave reasonably in a state of *isotropic deformation* as well as in a *plane stress state of equi-biaxial stretch*.‡ In the former case one has $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and readily finds from (2.9), (2.4) that $\tau_{11} = \tau_{22} = \tau_{33} = \tau$ where

$$\tau = 2\mu\left(\frac{9\mathscr{H}'(i_1)}{i_1^2} - 1\right), i_1 = 3\lambda. \tag{2.10}$$

Thus the Cauchy stress component τ is monotone increasing with λ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}i_1}\left(\frac{\mathscr{H}'(i_1)}{i_1^2}\right) > 0 \text{ for all } i_1 > 0. \tag{2.11}$$

On the other hand, in a state of plane stress equi-biaxial stretch one has $\lambda_1 = \lambda_2 = \lambda$, $\tau_{33} = \sigma_{33} = 0$. In this case eqns (2.9), (2.4) show that $\tau_{11} = \tau_{22} = \tau$ with

$$\tau = 2\mu\left(\frac{\lambda}{\lambda_3} - 1\right), \mathscr{H}'(2\lambda + \lambda_3) = \lambda^2. \tag{2.12}$$

In order that the material admit such a state of deformation it is therefore necessary and sufficient that $(2.12)_2$ yield a (positive) value for the transverse stretch λ_3 corresponding to a prescribed value of the in-plane stretch λ. We will assume that there exists such a root $\lambda_3 = \hat{\lambda}_3(\lambda) > 0$ of $(2.12)_2$ which is in fact unique, differentiable and monotone decreasing for $0 < \lambda < \infty$. Observe from the first of (2.12) that the Cauchy stress component τ is then a monotone increasing function of the stretch λ. One can show that such a root $\hat{\lambda}_3(\lambda)$ exists if and only if there is a number $i_0 \in (1, 3)$ such that

$$\mathscr{H}'(i_0) = 0, \frac{\mathscr{H}'(i_1)}{i_1^2} \to \frac{1}{4} \text{ as } i_1 \to \infty, \mathscr{H}''(i_1) > [\mathscr{H}'(i_1)]^{\frac{1}{2}} > 0 \text{ for } i_1 > i_0. \tag{2.13}$$

We omit the derivation of this result since it is entirely analogous to the corresponding analysis in the plane strain case ([5], Section 2). The remainder of this paper will be concerned with an *arbitrary harmonic material* subject to the requirements (2.8), (2.11), (2.13).

†The most general harmonic material introduced by John includes a term in W which is linear in the invariant i_2. As pointed out by John ([1], Section 6) however, energy densities without this term are of particular interest.
‡The latter state of deformation occurs locally on traction-free boundaries in the problem of concern here.

3. PRESSURIZED HOLLOW SPHERE: DEFORMATION AND STRESS FIELDS

Suppose now that the region $\mathscr{R}_0$ occupied by the undeformed body is a hollow sphere of internal radius a and external radius b and that the sphere is subjected to inner and outer pressures p_i, $p_0(\geq 0)$. The resulting (spherically symmetric) deformation is given by

$$R = f(r)r, (f(r) > 0), \Theta = \theta, \Phi = \phi \text{ on } \mathscr{R}_0, \tag{3.1}$$

where we have used spherical polar coordinates (r, θ, ϕ) and (R, Θ, Φ) to describe the location of a particle in the undeformed and deformed configurations respectively.

The spherical components of the deformation gradient tensor are found (see, e.g. [9]), using (2.1), (3.1), to be

$$F_{rr} = f(r) + rf'(r), F_{\theta\theta} = F_{\phi\phi} = f(r), \tag{3.2}$$

with the remaining components of $\mathbf{F}$ being zero. In view of (2.3), (3.2), the principal invariants i_1, i_3 are then given by

$$i_1 = 3f(r) + rf'(r), i_3 = f^2(r)[f(r) + rf'(r)]. \tag{3.3}$$

Since the deformation gradient tensor $\mathbf{F}$ here happens to be symmetric, the rotation tensor $\mathbf{R}$ in the polar decomposition (2.2) is the identity tensor. The spherical components of Cauchy stress are thus given by (2.4), (2.9), (2.1) as

$$\tau_{RR} = 2\mu\left(\frac{\mathscr{H}'(i_1)}{f^2(r)} - 1\right), \tau_{\Theta\Theta} = \tau_{\Phi\Phi} = 2\mu\left(\frac{\mathscr{H}'(i_1)}{f(r)[f(r) + rf'(r)]} - 1\right), \tag{3.4}$$

on $\mathscr{R}_0$ with i_1 being given by $(3.3)_1$. The shear components of stress are zero. The equilibrium eqns (2.5) in this case reduce to ([9])

$$\frac{d}{dR}\tau_{RR} + \frac{2}{R}(\tau_{RR} - \tau_{\Theta\Theta}) = 0 \text{ for } af(a) < R < bf(b), \tag{3.5}$$

which, on using eqns (3.1) and (3.4), yields

$$\frac{d}{dr}[\mathscr{H}'(i_1)] = 0 \text{ for } a < r < b. \tag{3.6}$$

Equation (3.6) shows that $\mathscr{H}'(i_1)$ is constant on (a, b). In view of the assumed monotonicity† of $\mathscr{H}'(i_1)((2.13)_3)$ this in turn implies that *the invariant* i_1 *is constant throughout the body*,

$$i_1 = rf'(r) + 3f(r) = c_1 \text{ for } a < r < b. \tag{3.7}$$

Upon integration, (3.7) gives

$$f(r) = \frac{c_1}{3} + \frac{c_2}{r^3}, a < r < b, \tag{3.8}$$

where c_2 is an arbitrary constant.

Turning next to the boundary conditions of the problem, we have

$$\tau_{RR} = -p_i \text{ at } R = af(a), \tau_{RR} = -p_0 \text{ at } R = bf(b), \tag{3.9}$$

†Actually, one needs to tentatively assume here that $\mathscr{H}'(i_1)$ is monotone on the entire interval $0 < i_1 < \infty$. It will subsequently become apparent that i_1 is greater than i_0 so that $(2.13)_3$ in fact suffices.

which, in view of (3.1), (3.4), (3.7), are equivalent to

$$2\mu\mathscr{H}'(c_1) = (2\mu - p_i)f^2(a) = (2\mu - p_0)f^2(b). \tag{3.10}$$

We will restrict attention to the case in which the *applied pressures are both less than* 2μ. It will be seen, subsequently, that pressures outside of this range are not of physical interest. Equations (3.10), (2.11), (2.13) now show that

$$\mathscr{H}'(c_1) > 0, c_1 > i_0 > 0. \tag{3.11}$$

Substituting (3.8) into (3.10) results in two algebraic equations involving the constants c_1, c_2 which may be simplified to read

$$\frac{9\mathscr{H}'(c_1)}{c_1^{\,2}} = \frac{(b^3 - a^3)^2(2\mu - p_0)(2\mu - p_i)}{2\mu(b^3\sqrt{2\mu - p_i} - a^3\sqrt{2\mu - p_0})^2},$$
$$3\frac{c_2}{c_1} = \left(\frac{\sqrt{2\mu - p_i} - \sqrt{2\mu - p_0}}{a^3\sqrt{2\mu - p_0} - b^3\sqrt{2\mu - p_i}}\right)a^3b^3. \tag{3.12}$$

A detailed analysis of the foregoing equations is postponed until the following section where we will show that, for any harmonic material satisfying (2.8), (2.11), (2.13), eqns (3.12) can be solved for unique values of the constants c_1, c_2 provided that the applied pressures lie in a certain range.

Expressions for the associated true stress components are found from (3.4), (3.8), (3.12) to be

$$\tau_{RR} = 2\mu\left(\frac{Ar^6 - 2Br^3 - 1}{B^2r^6 + 2Br^3 + 1}\right) = 2\mu\frac{Ar^6 - 2Br^3 - 1}{(Br^3 + 1)^2},$$
$$\tau_h \equiv \tau_{\Phi\Phi} = \tau_{\Theta\Theta} = 2\mu\left(\frac{Ar^6 + Br^3 + 2}{B^2r^6 - Br^3 - 2}\right) = 2\mu\frac{Ar^6 + Br^3 + 2}{(Br^3 + 1)(Br^3 - 2)}, \tag{3.13}$$

where

$$A = [9\mathscr{H}'(c_1)/c_1^{\,2} - 1]B^2, B = c_1/3c_2. \tag{3.14}$$

In the particular case when the applied pressures are small, ($p_i/2\mu$, $p_0/2\mu \ll 1$), it is not difficult to show that upon linearization (3.12)–(3.14) yield the classical results (see, e.g. Section 94 of [8]) according to the infinitesimal theory of elasticity

$$\tau_{RR} = \left(2\mu\frac{A}{B^2}\right) - \left(\frac{4\mu}{B}\right)\frac{1}{r^3}, \tau_h = \left(2\mu\frac{A}{B^2}\right) + \left(\frac{2\mu}{B}\right)\frac{1}{r^3},$$
$$2\mu\frac{A}{B^2} = \frac{p_ia^3 - p_0b^3}{b^3 - a^3}, \frac{B}{4\mu} = \frac{(b^3 - a^3)}{a^3b^3(p_i - p_0)}. \tag{3.15}$$

4. RESULTS AND DISCUSSION

We now examine some features of the results derived in the previous section. Observe first that according to (3.12) the constants A, B in (3.14) do not depend on the constitutive function $\mathscr{H}$. It therefore follows from (3.13) that (within the class of harmonic materials) the true stress *distribution in the sphere, expressed as a function of the undeformed coordinates, depends on the material at most through its infinitesimal shear modulus* μ. We emphasize that all of the particular properties that we now proceed to study also bear this same material independence.

4.1 *Internal pressure case* ($p_0 = 0, 0 < p_i < 2\mu$)

The hoop stress τ_h at the inner surface, according to (3.13), is given by

$$\tau_h = 2\mu \frac{Aa^6 + Ba^3 + 2}{(Ba^3 + 1)(Ba^3 - 2)} \text{ at } r = a. \tag{4.1}$$

It follows that if $Ba^3 = 2$ at some value of the applied pressure (say p_∞), the *hoop stress at the inner surface becomes unbounded.*† On using (3.12) and (3.14), this condition can be solved for the critical pressure $p_\infty (< 2\mu)$, giving

$$p_\infty = (2\mu/9)(1 - a^3/b^3)(5 + a^3/b^3). \tag{4.2}$$

In the remainder of Section 4.1 we will restrict attention to values of pressure in the interval $0 < p_i < p_\infty$. One can verify from (3.12), (3.14) that in this case A and B are both bounded and also that $A < 0$, $B > 0$, $Br^3 - 2 > 0$ and $Ar^6 + Br^3 + 2 > 0$ for $a < r < b$. Thus, the hoop and radial stress components (3.13) remain bounded and the hoop stress is tensile.

Next, differentiating the second of (3.13) and using (3.14) leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\tau_h = -54 \frac{\mathscr{H}'(c_1)}{c_1^{\,2}} \frac{B^2 r^5 (Br^3 + 4)}{(B^2 r^6 - Br^3 - 2)^2}. \tag{4.3}$$

Since $Br^3 + 1 > 0$ for $a < r < b$, it follows from (4.3) and (3.11) that *the hoop stress* τ_h *decreases monotonically with the radius* r and achieves its maximum value at the inner boundary $r = a$.

We now turn to the question of existence of the solution formally derived in Section 3. This would be ensured by the existence of a positive solution c_1 of $(3.12)_1$. It is not difficult to verify that the inequalities

$$0 < \frac{(b^3 - a^3)(1 - p_i/2\mu)^{\frac{1}{2}}}{b^3(1 - p_i/2\mu)^{\frac{1}{2}} - a^3} < \frac{3}{2} \tag{4.4}$$

hold provided that $0 < p_i < p_\infty$. Hence, it follows that the right-hand side of $(3.12)_1$ (with $p_0 = 0$) is a number in the interval (0, 9/4). Thus, in view of (2.11), (2.13) we conclude that $(3.12)_1$ *can* indeed be solved for a *unique* value of $c_1 (> i_0 > 0)$.

It is interesting to specialize the present results to the case of a shell which is thin in its deformed configuration. An appropriate expression for the hoop stress is most easily derived by first integrating the equilibrium equation (3.5) through the thickness which, on using (3.9), leads to

$$\int_{af(a)}^{bf(b)} 2R\tau_{\Theta\Theta}\,\mathrm{d}R = p_i a^2 f^2(a). \tag{4.5}$$

In view of the presumed small thickness, one can approximate the integral in (4.5) by using the mean-value theorem. This, together with (3.10) yields the approximate result

$$\tau_{\Theta\Theta} \simeq \frac{p_i}{2} \frac{a}{b\sqrt{1 - p_i/2\mu} - a}, \tag{4.6}$$

for the (true) hoop stress in a *thin shell.* At small values of the applied pressure (4.6) reduces to the classical result $p_i a/2(b - a)$.

Finally we observe that the unboundedness of the hoop stress is associated with the well-known poor behavior of harmonic materials under compression. Note from (3.8), (3.12), (4.2) that the radial stretch at the inner wall $\lambda_r = F_{rr} = af'(a) + f(a)$ tends to zero as $p_i \to p_\infty$. This in turn leads to the vanishing of the Jacobian J and so, by (3.4) the hoop

†The numerator of (4.1) does not vanish when $Ba^3 = 2$. See (3.14), (3.12), (3.11).

stress becomes unbounded. The occurrence of such a singular stress behavior is nevertheless worth noting, since it is induced by a constitutive effect rather than by a load or geometric discontinuity.

4.2 *External pressure case* ($p_i = 0,\ 0 < p_0 < 2\mu$)

Again, the existence of a solution is guaranteed provided that the right-hand side of $(3.12)_1$ (with $p_i = 0$) is a number in the interval (0,9/4). This can be readily shown to be true since $p_0 < 2\mu$ and $b > a$. Furthermore, it is readily seen that $B < 0$ and $Br^3 + 1 < 0$ for $a < r < b$ whence the stress components τ_{RR} and τ_h remain bounded. It is straightforward to verify that $A < 0$ and that $Ar^6 + Br^3 + 2 < 0$ for $a < r < b$. Thus (3.13) shows that the hoop stress τ_h is always compressive.

It follows from (3.12), (2.13), (2.11) that $c_1 \to i_0$ and $c_2 \to -a^3 i_0/3$ as the applied pressure p_0 approaches the value 2μ. Consequently (3.8) shows that the deformed inner radius $af(a) \to 0$ in this limit thus implying that *the cavity closes* as $p_0 \to 2\mu$. (Note from (3.8), (3.12) that the hoop stretch $f(r)$, and hence the Jacobian J, vanishes at the inner boundary as $p_0 \to 2\mu$.)

We now show that in contrast to the corresponding linear problem the (numerically) largest hoop stress in the sphere does *not* always occur at the cavity. Differentiating $(3.13)_2$ with respect to r and using (3.14) again yields (4.3) where B is now given by $(3.14)_2$, $(3.12)_2$ with $p_i = 0$. The monotonicity of τ_h is thus seen to depend on the sign of $Br^3 + 4$. In particular if $Br^3 + 4 < 0$ for $a < r < b$, then $|\tau_h|$ decreases with r and its largest value occurs at the inner wall. Similarly, if $Br^3 + 4 > 0$ on the entire interval $a < r < b$, the largest value of $|\tau_h|$ occurs at the outer wall. On the other hand, if $Br^3 + 4 > 0$ for $a < r < r_m$, $Br_m^3 + 4 = 0$ and $Br^3 + 4 < 0$ for $r_m < r < b$ (for some $r_m \epsilon (a, b)$), the greatest value of $|\tau_h|$ occurs at the radius r_m. On using the explicit expression for B these conditions may be examined in detail. We omit the algebraic details of this straightforward but lengthy calculation and simply record its results: ($t = (a/b)^3$).

(i) *When*

$$\frac{p_0}{2\mu} \le \frac{(1-t)(7-t)}{(4-t)^2},$$

$|\tau|_{max}$ *always occurs at* $r = a$. For larger values of $p_0/2\mu$, two cases must be considered:

(ii) *Case* (a): $t > \frac{1}{4}$. *When*

$$\frac{(1-t)(7-t)}{(4-t)^2} < \frac{p_0}{2\mu} < \frac{(1-t)(7t-1)}{9t^2},$$

$|\tau_h|_{max}$ *occurs at* $r = r_m \equiv (-4/B)^{1/3}$ *and when*

$$\frac{(1-t)(7t-1)}{9t^2} < \frac{p_0}{2\mu} < 1,$$

$|\tau_h|_{max}$ *occurs at* $r = b$. *The location of this maximum stress monotonically moves outwards as the pressure is increased.*

Case (b): $t < \frac{1}{4}$. *When*

$$\frac{(1-t)(7-t)}{(4-t)^2} < \frac{p_0}{2\mu} < 1,$$

$|\tau_h|_{max}$ *occurs at* $r = r_m \equiv (-4/B)^{1/3}$, *which is always an interior point (even as* $p_0 \to 2\mu$*)*

The explicit value of the *largest hoop stress* may now be calculated using (3.12), $(3.13)_2$, (3.14) and the preceding results. In the special case when the cavity is small, or equivalently, for a cavity in an infinite medium, ($a/b \to 0$, that is, $t \to 0$), result (i) above shows that $|\tau_h|_{max}$ is located at the cavity wall provided $p_0/2\mu \leq 7/16$; its value is given by

$$\frac{|\tau_h|_{max}}{p_0} = 3\left\{1 + \frac{p_0}{\mu} + \left(1 - \frac{p_0}{2\mu}\right)^{1/2}\right\}^{-1}. \tag{4.7}$$

On the other hand when $p_0/2\mu$ exceeds 7/16, result (ii), case (*b*) shows that the maximum hoop stress occurs at a radius $r_m \epsilon (a, b)$, where

$$\frac{r_m}{a} = \left\{4\frac{p_0}{2\mu}\left(1 + \sqrt{1 - \frac{p_0}{2\mu}}\right)^{-1}\right\}^{1/3}, \tag{4.8}$$

and its value is

$$\frac{|\tau_h|_{max}}{p_0} = \left(1 + \frac{8p_0}{2\mu}\right)\bigg/\left(\frac{9p_0}{2\mu}\right). \tag{4.9}$$

Thus (4.7), (4.9) yield values for the *stress concentration factor* $K = |\tau_h|_{max}/p_0$. A graph of K versus $p_0/2\mu$ is sketched in Fig. 1.

On the other hand, the *hoop stress at the cavity* (as $t \to 0$) is given by the r.h.s. of (4.7) for all values of the applied pressure $0 < p_0 < 2\mu$. The variation of this stress with pressure p_0 is also shown in Fig. 1.

4.3 *Combined pressure case* ($0 < p_i < 2\mu$, $0 < p_0 < 2\mu$)

The analysis of this case is very similar to that of the preceding cases. We will merely record the principal results.

In the case when $p_0 < p_i$ the greatest hoop stress occurs at the inner wall. Moreover,

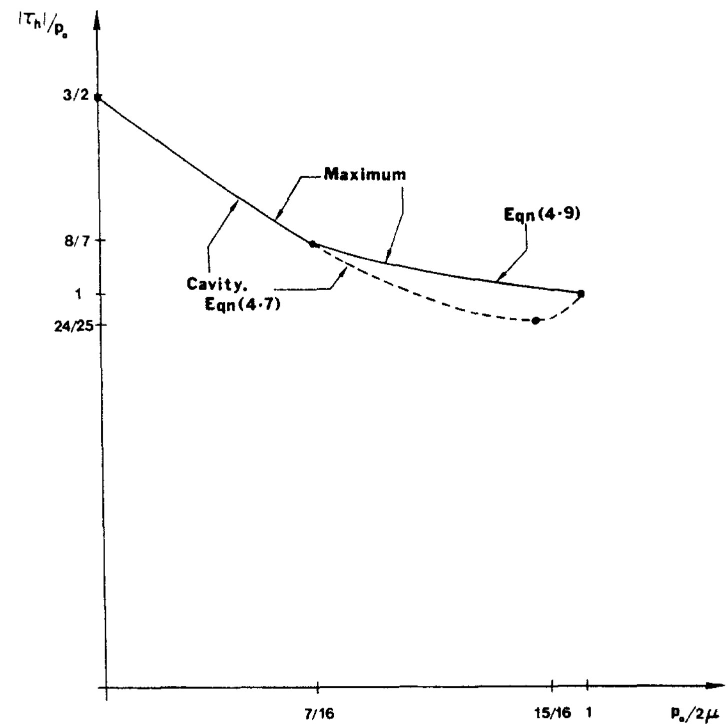

Fig. 1. Hoop stress for externally pressurized hollow sphere ($a \ll b$). Solid curve denotes $|\tau_h|_{max}/p_0$, dashed curve denotes $|\tau_h|_{r=a}/p_0$. Solid and dashed curves coincide for $0 \leq p_0/2\mu \leq 7/16$.

this value becomes unbounded when

$$3b^3(2\mu - p_i)^{1/2} = (a^3 + 2b^3)(2\mu - p_0)^{1/2} \tag{4.10}$$

which corresponds to the "critical condition" in the present situation. It is necessary, therefore, to further restrict the values of the applied pressures here to the range

$$(2\mu - p_i)^{1/2}/(2\mu - p_0)^{1/2} > (a^3 + 2b^3)/3b^3. \tag{4.11}$$

In the event that (4.11) holds, the existence of a solution is guaranteed as before.

When $p_i < p_0$ the stress components remain bounded, a solution exists and the cavity closes in the limit $p_0 \to 2\mu$ at fixed p_i.

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